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Force-linear mechanisms

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Abstract

This paper presents a mathematical model of a large class of machines which experience internal friction losses and maintain an essentially linear relation between input and output forces. The model is described in a general setting and a universal theorem is derived quantifying net work done and energy lost over a given process. Examples are given to illustrate the application of the model to predicting the performance of reciprocating heat engines equipped with such mechanisms.

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1. Introduction

Force-linear mechanisms are machines in which the relation between input and output (or dual input) force or torque is as linear as it can be. Many real machines are exactly force-linear, ranging from simple harmonic speed reducers [1] to the crank-connecting rod mechanisms described below. Other machines can be sufficiently well approximated as force-linear, or as serial connections of such. This paper introduces a general mathematical model for force-linear machines and derives a formula for net work output in terms of a given input process. The formula classifies and quantifies how, when, and where these machines suffer frictional losses.

The kind of insight this yields can be useful in designing and applying machines for specific tasks. To illustrate, the paper applies the force-linear results to reciprocating heat engines. It is shown first how the model can be used to evaluate the match between a mechanism and a thermodynamic cycle. A further application of the theory is the determination of the response of a large class of engines to pressurization or supercharging. Conditions are derived under which engine performance is improved by supercharging. At this point a counter-intuitive result is revealed and a specific example is constructed in which engine performance is actually degraded by supercharging.

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2. Machines

Fig. 1 schematically represents a basic 2-actuator machine. The machine, whose internals are not subject to analysis here, interacts with external entities through its actuators. Each actuator is a device capable of either linear or angular displacement, but not both, and capable of accepting or applying force or torque respectively. Coordinate systems, represented by x and y , are assigned to the range of motion of each actuator. The kinematic connection between the actuators is mathematically expressed with reference to a single parameter s . This is a parameter in terms of which the position of all of the machine elements can be specified. For example, s may be taken to be the crank angle in an analysis of a 2-stroke IC engine, or taken to be camshaft angle in a 4-stroke. In this paper, $x(s)$ will denote the piston position and $y(s)$ output shaft position when applied to engines. The operation or motion of a machine is represented by the time variation of the state parameter.

2.1. Constrained motion

For the purpose of analyzing machine operation in this paper, its motion in time is taken as fixed. The most common assumption for example in the analysis of an engine would be that its shaft has a uniform rotational motion. This is satisfied close enough if the flywheel is relatively large or there are a number of cylinders working on the same crankshaft. Motions other than uniform rotary can be handled as well, but the motion of the device undergoing analysis, whatever it is, is a priori specified. Thus an underlying assumption in any instance of analysis in this paper is that the parameter s changes in time in a prescribed way, that is, that a functional relation $s = \sigma(t)$ is given.

The advantage of this *constrained motion analysis* is that it reduces the rest of the machine description to a relation solely between the actuator forces and the state parameter. With the motion in time of the machine state parameter specified, the motion of every part of the machine in time is set by its kinematic geometry. Taking the masses of the internal machine members into account, the force relation between f and g at a given state parameter value s results from a static type force analysis with joint friction and any appropriate inertia forces and torques included. Thus constrained motion analysis allows one to work with a force relation of the simple form $F(s, f, g) = 0$.

It is important to note that *if the operating regime $s = \sigma(t)$ changes, the force relation $F(s, f, g) = 0$ will change also*. A given force relation remains in effect only as long as the motion giving rise to it remains the same. Note also that the machine state parameter can be taken to actually be

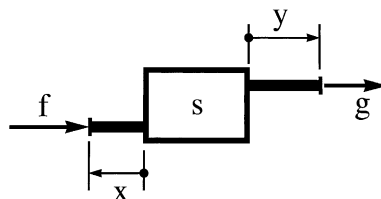


Fig. 1. Conceptual representation of a two-actuator machine.

clock time, $s = t$, but then one must be especially careful to remember that a fixed motion of the machine is assumed. A change of motion would entail a new parameterization of the machine. As long as this is kept in mind, it is actually simpler to use t as the state parameter [4]. A machine with a prescribed motion will be denoted by $M = [x(s), y(s), F(s, f, g)]$.

2.2. Sign conventions

For uniformity, energy exchanges will be expressed relative to the mechanism. This means that although the orientation of the actuator scales can be arbitrarily chosen, once chosen, *the positive external force or torque direction is taken to be in the direction of increasing actuator position coordinate*. For simplicity it will be henceforth assumed in this paper that the motion of the mechanism is such that the state parameter s increases in time. There is no loss of applicability here, since a simple reparameterization can always be made to meet this requirement.

Having $\sigma'(t) = \dot{s} > 0$ allows a straightforward identification of the direction of work flow. If f represents the force or torque applied to the x -actuator, when $f \cdot x'(s) > 0$, the force f is doing work on the machine at state s . If $f \cdot x'(s) < 0$, then the machine is doing work against the external resistance f at state s .

2.3. Friction

In its most basic role, the mechanism of an engine acts as a work transducer. It only exchanges energy with the rest of the engine in the form of mechanical work transmitted through the motion of its actuators and through the dissipation of some of this work in the form of heat of friction lost to the surroundings.

2.4. Energy storage

A secondary function performed by the mechanism of an engine is that of an energy store. This is an incidental capacity, arising primarily by virtue of the fact that machines have moving parts with mass, and so are capable of storing and releasing kinetic energy. Mechanisms can also involve compression of springs or the like, and so can also store and release potential energy internally. However, in typical engines, the internal energy content of the mechanism section is small compared to the energy stored in the flywheel, and the energy fluctuation of the internal members of an engine mechanism can usually be safely ignored. This state of affairs will be modeled here by postulating that the mechanism proper maintains a constant internal mechanical energy, that is the total kinetic and potential energy within the mechanism is constant throughout its operation.

2.5. Law of the machine

The first law of thermodynamics applied to the operation of a machine as modeled above requires that the work done by the machine never exceeds that done to it. The instantaneous expression of this is:

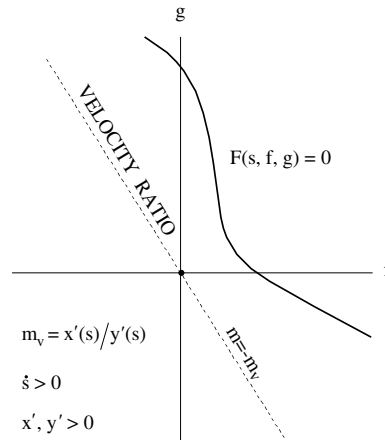


Fig. 2. The force relation of a machine at a state s for which $x'(s)$ and $y'(s)$ are both positive.

$$\text{if } F(s, f, g) = 0 \text{ and } \dot{s} > 0 \text{ then } fx'(s) + gy'(s) \geq 0 \quad (1)$$

Fig. 2 depicts this condition. A hypothetical force relation $F(s, f, g) = 0$ is shown in the f – g plane for a particular value of s . Condition (1) is satisfied in as much as the graph of F is in one of the closed half-planes determined by the line $g = -fx'(s)/y'(s)$. The graph is in the upper half-plane if $y'(s)$ is positive and in the lower if negative. Recall that $\dot{s} > 0$ is assumed here; inequality (1) has the opposite sense when $\dot{s} < 0$.

The slope $m_v = x'(s)/y'(s)$ is called the *velocity ratio* at state s . In a reciprocating engine, piston velocity relative to s , namely $x'(s)$, alternates in sign while the output shaft typically has unidirectional velocity which can be taken to be positive. For such a machine, the force relation will always lie above the velocity ratio line, $g = -m_v f$.

2.6. Force processes

Flywheel equipped engines invariably have mechanisms with a special *non-jamming* characteristic. Whatever force (within a reasonable range anyway) the piston may apply to the mechanism in any state, there must, in principle at least, be a corresponding torque which the flywheel could apply to sustain the operation of the engine. This means that for every state s and force f , there is a force g such that $F(s, f, g) = 0$. Typically g is uniquely determined and thus in applications to engines and most machines one may regard g as functionally dependent upon s and f .

3. Force-linear machines

A machine operating with a given motion will be called *force-linear* or a *v-machine* if for each state s , the graph of $F(s, f, g) = 0$ in the f – g plane is the union of two coterminal rays. Fig. 3 illustrates the force relation of a *v-machine* at a state where $x'(s)$ and $y'(s)$ are positive.

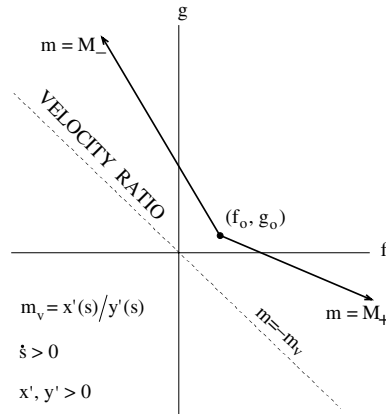


Fig. 3. The force relation of a v -machine where $x'(s), y'(s) > 0$.

Because of the restriction in this paper to non-jamming mechanisms, there is always a left and a right-hand ray for v -machines. The slopes of these rays will be denoted by M_- and M_+ respectively. These slopes are functions of the state s , and are typically piecewise continuous in s . The vertex $(f_o, g_o) = (f_o(s), g_o(s))$ is also a piecewise continuous function of s . Note that for force-linear machines, the first law (1) can be written as

$$(f - f_o)x'(s) + (g - g_o)y'(s) \geq 0 \quad (2)$$

where $\dot{s} > 0$ is being assumed throughout as stated.

A compact expression for a the force relation of a non-jamming force-linear machine is the following

$$g - g_o = M_+(f - f_o)^+ - M_-(f - f_o)^- \quad (3)$$

The positive and negative part functions used here are as in standard mathematical practice [2] namely:

$$z^+ = \max\{0, z\} \quad \text{and} \quad z^- = \max\{0, -z\}$$

Note that both parts are non-negative by definition and that the following is an identity:

$$z = z^+ - z^-.$$

A common force-linear mechanism is the crank-connecting rod system shown in Fig. 4. With pure Coulomb friction on the main shaft, the crankpin, and the piston slider, the relation between

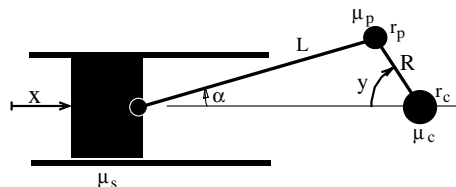


Fig. 4. The crank-connecting rod mechanism.

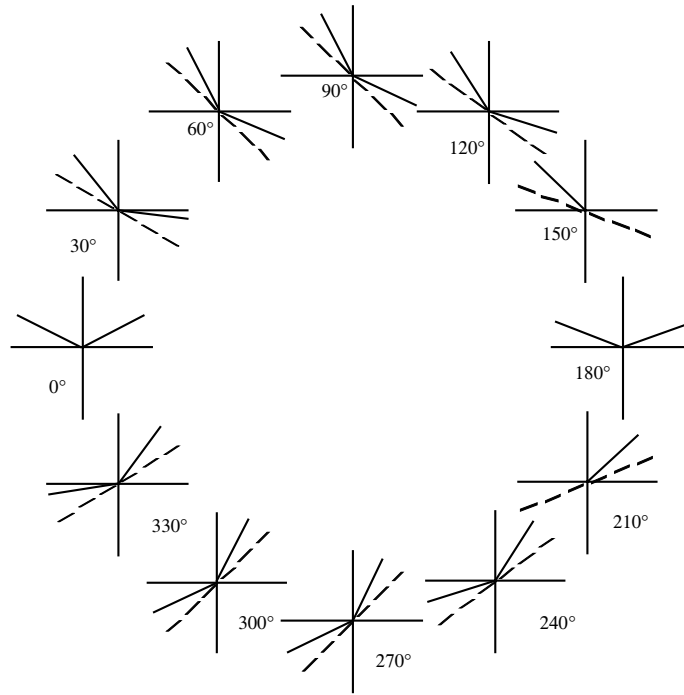


Fig. 5. The force relation of a crank-connecting rod mechanism shown for every 30° of crank rotation. The horizontal axis represents the net force acting on the piston and the vertical axis represents the corresponding torque on the main shaft. Velocity ratio is represented by a dotted line.

the force f acting on the piston and the torque g applied to the crankshaft is force-linear. Fig. 5 shows the force relation of this mechanism for every 30° increment of crank rotation; large friction coefficients μ and large journal radii r were chosen in this example to more emphatically show the nature of the resulting force relation. The values of the parameters used in this example

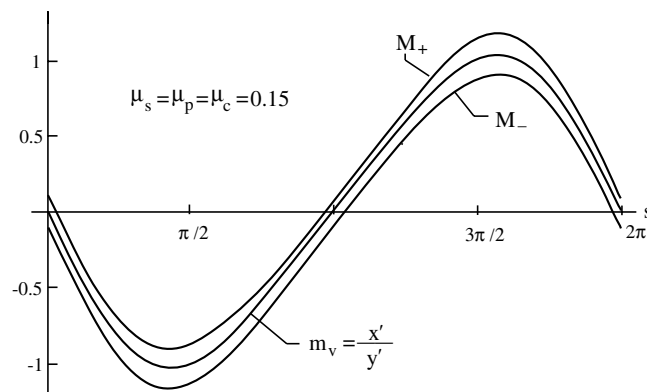


Fig. 6. Graphs of M_+ and M_- for the crank-connecting rod mechanism with friction coefficients set at 0.15. The central curve is the velocity ratio. With no friction, M_+ and M_- would coincide with the velocity ratio m_v .

are: friction coefficients $\mu_s = \mu_p = \mu_c = 1.0$; connecting rod to crank throw ratio $L/R = 4$; and journal radius to crank throw ratios $r_c/R = 0.32$ and $r_p/R = 0.28$.

For smaller friction coefficients, the graphs are similar but the v -arms are closer to the velocity ratio line. If constant preloading is added to the piston or the main crankshaft, the relation remains force-linear but the vertex is moved away from the velocity ratio line as generically represented in Fig. 3. Fig. 6 shows plots of the slopes M_+ and M_- vs. crankangle s for the same mechanism but with the friction coefficients reduced to 0.15.

In the following sections, a formula is derived for the shaft work output of a force-linear machine when subjected to a given force process. The form of the terms in the formula make it simple to identify and study the nature and magnitude of the friction losses incurred.

4. Kinematic power ratio

It is conceptually helpful to identify the ratio of net forces on the actuators normalized by the velocity ratio. This will be referred to as the *kinematic power ratio*. For force-linear machines there are two such ratios, denoted by k_+ and k_- , which correspond to the two rays of the v -relation. The kinematic power ratios for a force-linear machine are defined by

$$k_\alpha = \begin{cases} -M_\alpha y'/x' & \text{if } x' \neq 0 \\ 0 & \text{if } x' = 0 \end{cases} \quad (4)$$

where $\alpha \in \{+, -\}$. The kinematic power ratio is the ratio of the instantaneous power being done by the y -actuator to the power being delivered to the x -actuator. Note the minus sign in the definition. For applications to engines for example, if the x -actuator is the piston and the y -actuator is the output shaft, then the kinematic power ratios as defined by (4) represent the power out of the shaft divided by the power delivered to the piston.

The ratios k_+ and k_- are functions of the state parameter s and can vary over the entire range of real numbers. Fig. 7 shows plots of k_+ and k_- for the crank-connecting rod mechanism of Fig. 6. Fig. 8 shows k_+ and k_- for the same mechanism with all the friction coefficients increased to 1.0.

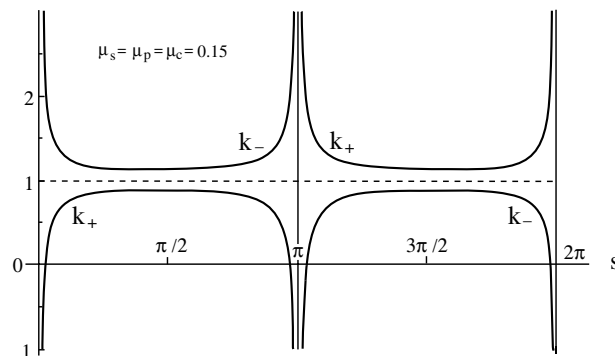


Fig. 7. Graphs of k_+ and k_- for the crank-connecting rod mechanism of Fig. 6 with friction coefficients of 0.15.

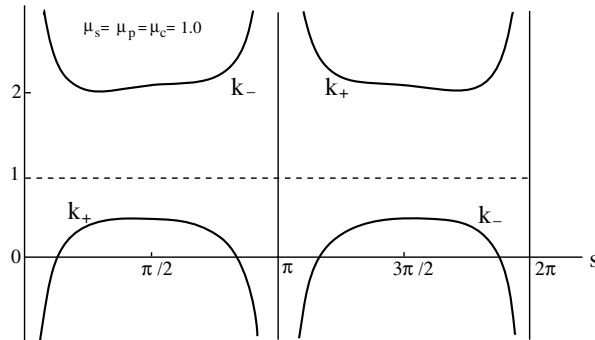


Fig. 8. Graphs of k_+ and k_- for the crank-connecting rod mechanism as in Fig. 7 with the friction coefficients increased to 1.0.

5. The work formula for force-linear machines

Practical machines are evaluated by comparing the total energy exchanges between the two actuators of the machine performing a given task. Work is put into one end of the machine and the interest is in how much one gets out the other side. This is obviously the case with devices such as jacks, hoists, and automotive transmissions where effort is put into one side and beneficial effect appears out of the other. But it is also relevant when work alternately goes both out of and into each side as is the case with most reciprocating heat engines. Work is put into the piston side of the engine mechanism by the thermodynamic working substance, but also can be taken out at different times during the cycle to perform compression processes for example. The same is true on the flywheel side of the engine mechanism. Work is alternately put into and taken out of the flywheel as the engine operates. The net work out of the flywheel side over a cycle is the primary interest for engines, but this is a function of how well or badly all the energy exchanges over a cycle are performed. A more precise general understanding of these exchanges is the goal of this paper.

As will be seen, the kinematic power ratios defined above are helpful in classifying and quantifying these energy exchanges for a given force process acting on a force-linear mechanism. Using these ratios, the theorem deduced below expresses the net work done by the y -actuator of a machine in terms of the work process applied to the x -actuator in such a way that the losses and gains can be clearly identified and easily understood.

Let $M = [x, y, F(s, f, g)]$ be a force-linear machine described by (3) and $\langle f, g \rangle$ a force process acting on it as s ranges over the closed bounded interval I , that is, f and g are functions of the state parameter s such that $F(s, f(s), g(s)) = 0$ for all s in I . It is assumed that all the functions here are piecewise continuous in s . Then the *work output* done by the y -actuator is

$$W_o\langle g, y \rangle = - \int_I g \, dy = - \int_I g(s) y'(s) \, ds$$

Let K be the union of all the closed subintervals of I in the interior of which $x'(s) \equiv 0$, and let J be the closure of the complement. Then J is a union of closed intervals and together with the intervals of K partition I with only endpoint overlapping. Thus

$$W_o\langle g, y \rangle = - \int_I (g - g_o) dy - \int_I g_o dy = - \int_J (g - g_o) dy - \int_K (g - g_o) dy - \int_I g_o dy \quad (5)$$

The last term of (5) is a characteristic of the machine and does not depend upon the applied force process $\langle f, g \rangle$; this quantity will be denoted by W_d :

$$W_d = \int_I g_o dy$$

The receding term in (5) arises from the portions of the working process when the x -actuator is stationary. This term represents a loss because when $x'(s) = 0$, relation (2) requires that $(g - g_o)y'(s) \geq 0$. This loss depends upon the applied force process, but in practical machines it is usually a minor effect. In heat engines it represents work that must be supplied from the flywheel to carry the engine over periods where the piston is motionless. In most engines, this occurs only in two instants during each cycle, in which case this loss is zero. This loss will be denoted by W_p :

$$W_p = \int_K (g - g_o) dy$$

Now the first integrand in (5) can be expanded by using the following calculation:

$$\begin{aligned} -(g - g_o)y' &= -[M_+(f - f_o)^+ - M_-(f - f_o)^-]y' \quad \text{by(3)} \\ &= [k_+(f - f_o)^+ - k_-(f - f_o)^-]x' \quad \text{by(4)} \end{aligned}$$

which holds when $x'(s) \neq 0$. Now using the identity $z = z^+ - z^-$ on x' , k_+ , and k_- , the last expression expands to

$$\begin{aligned} k_+^+(f - f_o)^+x'^+ - k_+^-(f - f_o)^+x'^+ - k_-^+(f - f_o)^-x'^+ + k_-^-(f - f_o)^-x'^+ + k_+^+(f - f_o)^-x'^- \\ - k_-^-(f - f_o)^-x'^- - k_+^+(f - f_o)^+x'^- + k_-^-(f - f_o)^+x'^- \end{aligned} \quad (6)$$

In this sum, the fourth and eighth terms are in fact zero under our assumption that $\dot{s} > 0$. The fourth term

$$k_-^-(f - f_o)^-x'^+$$

could be positive only where

$$k_- < 0, \quad f - f_o < 0, \text{ and } x' > 0.$$

If $y' > 0$, then the first and last inequalities imply $M_- > 0$, but this contradicts condition (2) when $f - f_o < 0$. If $y' < 0$, then $M_- < 0$ and again (2) would not hold. Therefore the term is zero. A similar analysis shows the last term

$$k_+^-(f - f_o)^+x'^-$$

is zero also for $\dot{s} > 0$.

In addition to assuming the functions x' , y' , M_+ and M_- are piecewise continuous on the state parameter interval $I = [a, b]$, it is also reasonable to assume that they are of *finite oscillation*, that is, the inverse image of any open interval is a union of finitely many intervals open in I . This is a mathematical technicality and does not diminish the set of practical mechanisms to which the result can be applied. The assumption is sufficient for a mathematical proof that all the terms in

(6) are integrable; this proof is given in the Appendix. Therewith we have derived the following result.

Theorem. *If $\langle f, g \rangle$ is a force process acting on the x and y actuators respectively of a force-linear machine described by (3) with $\dot{s} > 0$, then the net work done by the y -actuator is*

$$W_o\langle g, y \rangle = \int_I k_+^+(f - f_o)^+ dx^+ - \int_I k_+^-(f - f_o)^+ dx^+ - \int_I k_-^+(f - f_o)^- dx^+ \\ + \int_I k_-^-(f - f_o)^- dx^- - \int_I k_-^+(f - f_o)^+ dx^- - W_p - W_d \quad (7)$$

Formula (7) yields insight into exactly where and when the desired work and the undesired losses occur in the operation of a machine. The first three terms refer to motion in the direction of increasing x since they contain $dx^+ = x'(s)^+ ds$. The first of these terms

$$\int_I k_+^+(f - f_o)^+ dx^+$$

represents a positive contribution to $W_o\langle g, y \rangle$, the work output done through actuator y . These contributions occur exactly where $0 < k_+ \leq 1$ and where the net input force $f - f_o$ agrees in sign with x' . Such processes are called *efficacious*.

The next of the first three terms is

$$- \int_I k_+^-(f - f_o)^+ dx^+$$

Because the integrand is non-negative and a minus sign precedes the term, it represents a loss relative to the desired work $W_o\langle g, y \rangle$. These losses are incurred wherever k_+ is negative, $f \geq f_o$, and $x' > 0$. Here the net force on the x -actuator is in the same direction as its movement, but the value of k_+ is negative and so work is required from the y -side to assist or carry the machine through such situations. This kind of loss occurs for example near the “dead centers” in a crank-connecting rod mechanism. We will refer to this as a *dead center loss* although in general it can occur in other situations.

The last loss term of this group is

$$- \int_I k_-^+(f - f_o)^- dx^+$$

A loss of this type occurs when the net x -side force, $f - f_o$, opposes the motion of the x -actuator. We refer to such situations as *forced processes* since work is needed from the y -side in order to force the machine to move against the net input on the other side. Forced work processes occur in reciprocating engines whenever compression takes place above external pressure, or expansion is forced to take place below the external or buffer pressure [4]. Note that when forced work occurs, the corresponding kinematic work ratio k_- is greater than unity.

Interpretation of the three terms involving dx^- is similar. The term

$$\int_I k_-^+(f - f_o)^- dx^-$$

represents a gain in $W_o\langle g, y \rangle$ through an efficacious process in which $0 < k_- \leq 1$. The term

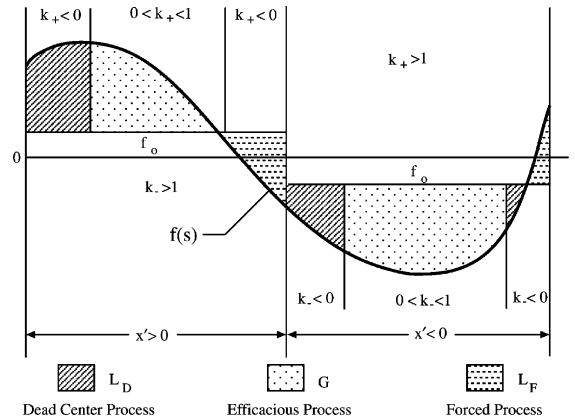


Fig. 9. Example of a force process $f(s)$ acting on the x -side of a force-linear machine with x -preload f_o .

$$- \int_I k_- (f - f_o)^- dx^-$$

represents losses occurring when the net input force agrees with actuator velocity but $k_- < 0$. This is another dead center type loss. Finally, the term

$$- \int_I k_+ (f - f_o)^+ dx^-$$

represents forced work; note $k_+ \geq 1$ when $x' < 0$.

These losses and gains are easily identified graphically. Fig. 9 shows a force process f as a function of state parameter s for a force linear machine with a constant preload f_o on the x -actuator. The regions which correspond to positive contributions to $W_o\langle g, y \rangle$ are identified by the label G . These are the efficacious portions of the force process. Note that the work contribution to $W_o\langle g, y \rangle$ is not the area of the corresponding region, but is given by the first and fourth terms of (7). The regions corresponding to dead center type losses are identified by L_D . The forced process loss regions are identified by L_F .

Because of the independence of k_x from f , there is a general monotonic relation between the area of a region of one specific type between f and f_o and the magnitude of the corresponding gain or loss term. To illustrate, just considering one region, say one of type L_D , if one were to change the force process f so that the new region was contained (in the set-theoretic sense) within the original, then the L_D loss for the new process would be smaller than for the original. This same type of monotonic relation holds for the other type regions as (7) makes clear.

6. Application to reciprocating engines

If an engine cycle with pressure p acts on a piston of area A attached to the x -actuator of the engine mechanism, and if p_o denotes the external or buffer pressure on the non-workspace side of the piston, the force process applied is $f = (p - p_o)A$ and $dx = x'(s)ds = (1/A)dV$. Thus the integral terms in (7) have the “ $p dV$ ” form

$$\int_I k_\alpha^\beta \left(p - p_o - \frac{f_o}{A} \right)^\alpha dV^\gamma \quad \text{where } \alpha, \beta, \gamma \in \{+, -\}$$

which directly relate to the pressure–volume diagram of the engine cycle. The first three terms of (7) in which $\gamma = +$ correspond to expansion processes. The three terms in which $\gamma = -$ relate to the compression processes of the cycle.

Fig. 10 shows a simple pressure–volume diagram of a heat engine with a constant external or buffer pressure p_o . Fig. 11 shows the corresponding force applied to the piston as a function of the

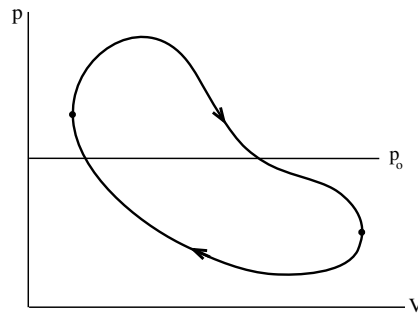


Fig. 10. Example of a simple engine p – V diagram with constant buffer pressure.

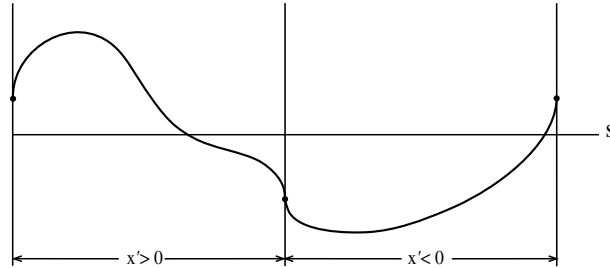


Fig. 11. Graph of piston force arising from the cycle of Fig. 10 vs. mechanism state parameter s .

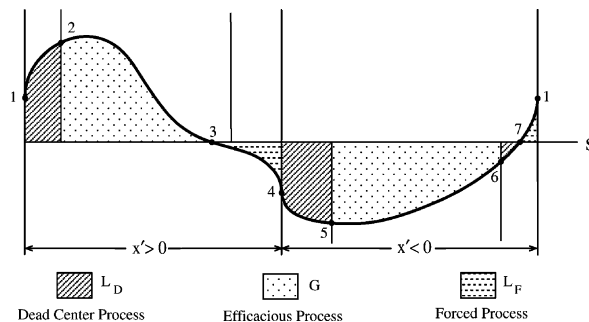


Fig. 12. The piston force process of Fig. 11 with the characteristics of a crank-connecting rod force-linear mechanism. In this example, there is no piston preloading.

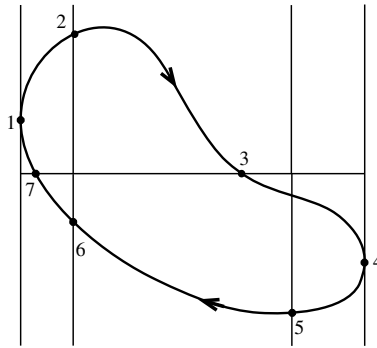


Fig. 13. The p - V diagram of Fig. 10 with the mechanism characteristics of Fig. 12 superimposed.

mechanism state parameter s . Fig. 12 imposes on this the characteristics of a force-linear mechanism typical of a crank-connecting rod system with no preloading. This diagram serves to identify how the engine cycle interacts with the engine mechanism. The portion of the cycle process from 1–2 produces a dead-center type loss. 2–3 is an efficacious process contributing work to the flywheel and output shaft. 3–4 is a forced expansion. 4–5 is more dead-center loss. 5–6 is an efficacious compression. 6–7 is another process incurring dead-center loss. Finally, 7–1 is a forced compression process. In simple cases such as this, most of the above information can be qualitatively displayed on a slightly modified p - V diagram as shown in Fig. 13.

7. Pressurization effects

As an illustration of the utility of the above results, we consider here the effects of pressurizing or supercharging an engine. Most engine cycles, at least the textbook idealizations of them, are *monomorphic* [3]. In a monomorphic engine, the workspace pressure function is proportional to the mass of the working fluid, everything else being the same. Examples of monomorphic cycles include the ideal Stirling, Carnot, and Otto with fixed temperature and volume extremes, more generally all similarly restricted Reitlinger and Crossley cycles, and many nameless others.

The indicated work of a monomorphic cycle is proportional to the workspace fluid mass. This is the ideal response to charging an engine with more working fluid. However what counts in practice is not just the indicated output, but what comes out of the shaft. This does not always improve with charging. We treat here two important modes of charging: system charging, and workspace charging with buffering from below.

7.1. System charging

In engines having an enclosed pressure-worthy crankcase or buffer chamber, it is possible to pressurize both the workspace and the bufferspace uniformly as a system, by increasing the mass of the gas in each space by the same factor. Here it will be assumed that the bufferspace is monomorphic along with the workspace. A monomorphic workspace pressure p can be represented mathematically by

$$p(s) = mp_1(s)$$

where m is the factor by which the mass of working fluid is increased in charging, and $p_1(s)$ is the pressure function at a base charge corresponding to a unit factor, i.e., $m = 1$. Likewise the buffer pressure is representable by

$$mp_o$$

where p_o is the external pressure at base charge $m = 1$. Note that p_o need not be constant, but is usually nearly so in practice [5].

The force applied to the piston at charge level m is therefore

$$f = m(p_1 - p_o)A.$$

Preloading on the piston, represented in our analysis by f_o , is typically small in a well-designed and constructed engine. Taking $f_o = 0$ as the best possible case, from (3) and (5) it follows that

$$W_o\langle g, y \rangle = - \int_I (g - g_o) dy - W_d = m \int_I (M_-(p_1 - p_o)^- - M_+(p_1 - p_o)^+) dy - W_d.$$

Recall that W_d is a mechanism characteristic independent of the engine cycle and hence independent of m in the present application. This proves the following result.

Theorem. *The cyclic shaft work output of a monomorphic engine with a force-linear mechanism and negligible piston preload varies linearly with its system charge level.*

In practice, W_d is relatively small, due only to preloads acting directly on the output shaft, such as contact friction from seals or constant speed viscous friction from lubricating or sealing fluid. In the limiting case when $W_d = 0$, shaft output is directly proportional to system charge level m and mechanical efficiency is constant.

7.2. Workspace charging with buffering from below

The case where buffer pressure is left unchanged and the workspace charge level is elevated is a case of considerable practical importance since many engines are not equipped with a pressure-worthy crankcase or bufferspace. For such engines the buffer is the atmosphere and charging the workspace is the only possibility for improving performance through pressurization.

Unlike system charging, workspace charging will not always result in an increase in cyclic shaft output. Once the charge level has been raised to the point where the entire cycle is above the buffer pressure level, shaft output will either ideally increase or decrease monotonically as the charge level is further increased.

To see this, suppose m_b is the charge level where the cycle is just buffered from below, so that $mp_1 > m_b p_1 \geq p_o$ for all $m > m_b$. Again regarding piston preload f_o as negligible, only the first, second, and sixth integral terms of (7) are possibly non-zero when $m > m_b$ because $[mp_1 - p_o]^- = 0$. Also taking $W_p = 0 = W_d$, shaft work is then

$$W_o\langle g, y \rangle = \int_I k_+^+(mp_1 - p_o)^+ dV^+ - \int_I k_+^-(mp_1 - p_o)^+ dV^+ - \int_I k_+^+(mp_1 - p_o)^+ dV^-$$

These terms correspond to efficacious work on expansion, dead center loss on expansion, and forced work on compression respectively. Since $[mp_1 - p_o]^+ = mp_1 - p_o$ when $m > m_b$, shaft work can be written more simply as

$$W_o\langle g, y \rangle = \int_I k_+(mp_1 - p_o) dV$$

This shows that shaft output will vary linearly with charge level in the range $[m_b, \infty)$. Whether $W_o\langle g, y \rangle$ increases or decreases as m is increased beyond m_b is determined by the sign of the integral

$$\int_I k_+ p_1 dV \quad (8)$$

Calculation of this integral provides a simple test of the response of an engine to charging above buffer pressure. Physically, the response depends upon the match between the mechanism characteristic k_+ and the base volume-pressure cycle (V, p_1) . A good match results in an engine in which workspace charging increases output; a poor match produces an engine in which supercharging degrades performance. Examples of engines exhibiting these behaviors are given below. In cases where the mechanism has simple constant effectiveness, much more can be deduced about an engine's response to workspace charging [3].

As a simple example of the effects of workspace charging, consider an ideal Stirling cycle coupled to a force-linear mechanism having characteristics similar to a crank-connecting rod system. The engine is mathematically described by the following equations:

$$x = \frac{1}{A} V(s)$$

$$V(s) = V_m + \frac{V_M - V_m}{2} (1 - \cos s)$$

$$y = s$$

$$M_+(s) = (a - \sin s)(V_M - V_m)/(2A)$$

$$M_-(s) = (-a - \sin s)(V_M - V_m)/(2A)$$

$$p_E = \frac{mRT_H}{V(s)} \quad \text{for } 0 \leq s \leq \pi$$

$$p_C = \frac{mRT_C}{V(s)} \quad \text{for } \pi \leq s \leq 2\pi$$

where

s = crank angle,

A = piston area,

$V(s)$ = instantaneous engine workspace volume,

V_m = minimum workspace volume,

V_M = maximum workspace volume,

T_H = maximum working fluid temperature,

T_C = minimum working fluid temperature,

m = mass of working fluid,

R = ideal gas constant,

p_E = pressure during isothermal expansion, p_C = pressure during isothermal compression, a = mechanism constant.

The mechanism in this example is mathematically simpler than the crank-connecting rod mechanism discussed above. Piston motion is sinusoidal. The constant a determines the effectiveness of the mechanism. The larger the value for a , the poorer the performance of the mechanism; $a = 0$ corresponds to the frictionless case. Fig. 14(a) and (b) show the graphs of the kinematic power ratios for $a = 0.1$ and 0.3 respectively. It can be seen that the force characteristics of this model are quite similar to the actual crank-connecting rod mechanism.

Fig. 15 shows the ideal Stirling cycle mathematically described above for the specific temperature and volume compression ratios

$$\tau = \frac{T_C}{T_H} = 0.5 \quad \text{and} \quad r = \frac{V_M}{V_m} = 1.5$$

and three working mass charge levels. At the lowest shown charge level, $m = 1.5$, the minimum cycle pressure just equals the external buffer pressure depicted by the horizontal line. As the charge level is increased, the cyclic indicated work increases proportionally. With the buffer pressure held constant, shaft work will increase or decrease according to whether the sign of integral (8) is positive or negative.

The base cycle corresponding to $m = 1$ has pressure function given by

$$p_1 = \begin{cases} \frac{RT_H}{V(s)} & 0 \leq s \leq \pi \\ \frac{RT_C}{V(s)} & \pi \leq s \leq 2\pi \end{cases}$$

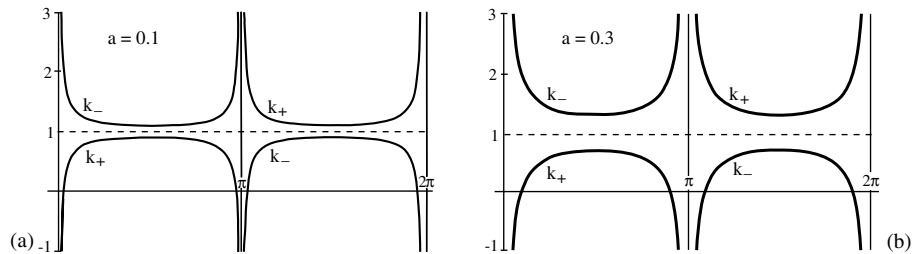


Fig. 14. Graphs of k_+ and k_- for a hypothetical force-linear mechanism with constant $a = 0.1$, and with constant $a = 0.3$.

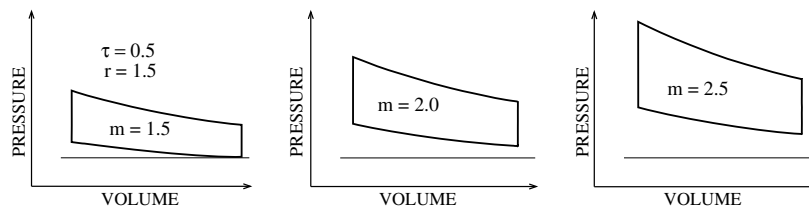


Fig. 15. An ideal Stirling cycle with fixed temperature and volume extremes, constant external buffer pressure, and increasing working fluid mass.

The mechanism power ratio k_+ is given by (4) which in this example is

$$k_+ = \left(1 - \frac{a}{\sin s}\right) \text{ for } s \neq 0, \pi \text{ or } 2\pi, \text{ and } k_+ = 0 \text{ at these points.}$$

Also, $V'(s) = \frac{V_M - V_m}{2} \sin s$. Thus

$$\int_I k_+ p_1 dV = \int_0^{2\pi} k_+(s) p_1(s) V'(s) ds = \frac{(r-1)RT_H}{2} \left[\frac{2(1-\tau) \ln r}{r-1} - \frac{a\pi(1+\tau)}{\sqrt{r}} \right]$$

The factor in the brackets above works out to 0.426 for the chosen cycle with $\tau = 0.5$ and $r = 1.5$ with the better mechanism having $a = 0.1$. This positive value means that shaft output will increase as workspace charge is increased. For the poorer mechanism having $a = 0.3$, the factor calculates to -0.343 which means that engine output will steadily decrease as the workspace charge is increased above buffer pressure.

8. Conclusion

The force-linear concept introduces a simple yet realistic model of machines appropriate for analysis under conditions of fixed or constrained motion. Applied to reciprocating engines, it permits the classification and quantification of the various frictional losses suffered by an engine in a way which improves intuition on how an engine cycle and a mechanism interact to determine overall engine performance. The force-linear model is also useful for determining the performance response of an engine to changes in its thermodynamic cycle, such as those due to supercharging.

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Appendix

Lemma. The functions $k_\alpha^\beta x^\gamma$ are piecewise continuous for all choices of $\alpha, \beta, \gamma \in \{+, -\}$.

Proof. From the assumptions of piecewise continuity and finite oscillation, it is possible to find a partition $P : a = s_0 < s_1 < s_2 < \dots < s_n = b$ of I into interior-disjoint subintervals $I_j = [s_{j-1}, s_j]$ with the following properties:

- the functions x', y', M_+ and M_- are continuous on the interior of each I_j and are continuously extendable to I_j ,
- one and only one of the following holds throughout the interior of each I_j :
 $x' > 0, \quad x' = 0, \quad \text{or} \quad x' < 0$
- k_+ and k_- are continuous on the interior of each I_j ,

- for each $\alpha \in \{+, -\}$, throughout the interior of each I_j , exactly one of the following holds:
 $k_\alpha < 0$, $0 \leq k_\alpha \leq 1$, or $1 < k_\alpha$.

Using this partition, it is easily seen that all the terms in expression (6) are piecewise continuous functions of s . On any of the subintervals I_j of the partition P , consider the function $k_\alpha(s)x'(s)^\gamma$ where $\alpha, \gamma \in \{+, -\}$. In a case where $x'(s)$ is identically zero on $\text{Int}(I_j)$ so also is $k_\alpha(s)x'(s)^\gamma$ zero on $\text{Int}(I_j)$. In the other cases, $x'(s)$ is of one sign throughout the interior of I_j . Therefore $x'(s)^\gamma$ is $x'(s)$, 0, or $-x'(s)$ throughout $\text{Int}(I_j)$. Hence x'^γ/x' is identically equal to one of the constant functions 1, 0, or -1 in $\text{Int}(I_j)$; denote this constant by c . Then

$$k_\alpha(s)x'(s)^\gamma = -M_\alpha(s)y'(s) \frac{x'(s)^\gamma}{x'(s)} = -M_\alpha(s)y'(s)c$$

in $\text{Int}(I_j)$. The right hand side of this equation is continuous in $\text{Int}(I_j)$ and the limits from the interior to the endpoints exist because they do for each factor. This proves that the functions $k_\alpha(s)x'(s)^\gamma$ are piecewise continuous on I for all choices of $\alpha, \gamma \in \{+, -\}$. From this the lemma follows because

$$[k_\alpha x'^\gamma]^+ = k_\alpha^+[x'^\gamma]^+ + k_\alpha^-[x'^\gamma]^- = k_\alpha^+ x'^\gamma$$

and

$$[k_\alpha x'^\gamma]^- = k_\alpha^+[x'^\gamma]^- + k_\alpha^-[x'^\gamma]^+ = k_\alpha^- x'^\gamma. \quad \square$$

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